Bridging Units: Resource Pocket 6

Number sequences

This pocket introduces the concepts of triangular, square and cube numbers and the terms 'arithmetic progression' and 'geometric progression'. It also formulates the ideas of Fibonacci and quadratic sequences. The current 2007 Key Stage 3 Programme of Study, covers some of these topics, stating the need to teach 'a range of sequences and functions based on simple rules and relationships'. Within the Attainment Target descriptors, linear and quadratic functions are mentioned, including finding the nth term.

Students will be used to the concept of number sequences, perhaps introduced through diagrammatical patterns of dots or objects. They are also likely to be familiar with 'spotting the pattern'. Due to the requirements of the new GCSE, it will be beneficial in Key Stage 3 to introduce the idea of number patterns being built up in different ways, which are not always linear. This will help to avoid an automatic assumption when studying GCSE that the nth term will be easily found and will be of the form $an + b$

All the content contained in this pocket forms the foundations of topics which are included on the GCSE Foundation tier. GCSE Basic Foundation content includes using square, cube, triangular and simple arithmetic sequences of numbers. The Additional Foundation content adds Fibonacci-type sequences, quadratic sequences and geometric progressions, where the common ratio is a positive rational number.

The GCSE specification requires term-to-term and position-to-term rules for sequences and an understanding of recursive sequences, so they will be included in introductory form here too.

This resource pocket progresses through three sections: developing understanding, skills builders and problem solving activities. As with all 9 resource pockets there are a number of different learning styles and approaches used to cater for a variety of learners.

1. Developing Understanding

These are class based, teacher led questions with suggested commentary to get the most from a class or small group discussion. The boxed text can either be copied onto the whiteboard for class discussion, or printed onto cards and handed out to students to be used for paired or small group work.

2. Skills Builders

These are standard progressive worksheets that can be used to drill core skills in a particular area. Skills Builder 2 could be adapted by removing the recursive formula column, and references to quadratic sequences, if preferred. Skills Builder 3 is more suitable for students progressing to the Higher tier at GCSE.

3. Problem Solving Activities

Extension activities for paired work or small group work to develop problem solving skills whilst focussing on a particular area of mathematics that students can learn to apply. Problem Solving Activity 2 is more suitable for students progressing to the Higher tier at GCSE.

Developing Understanding 1

Do you know any ‘famous’ number sequences?

Do they have special names?

How are the terms generated? Are they related to each other or to a special rule?

Display the information in the box on the board. Through discussion, the prior knowledge students have relating to well-known sequences will be exposed. Students could work in pairs or small groups and write down their answers on mini-whiteboards on their desk. Do not share the outcomes of the discussion as a whole class at this stage, but circulate around the groups and prompt students to think of sequences they have not yet thought of.

Hand out the set of cards below and ask students to cut them up and place them in pairs.

Triangular numbers	1, 8, 27, 64, ….
2, 4, 6, 8, ….	Even numbers
1, 1, 2, 3, 5, …	Cube numbers
Fibonacci numbers	1, 4, 9, 16, ….
Square numbers	Odd numbers
1, 3, 6, 10, ….	1, 3, 5, 7, ….

Discuss the pairings as a class. How many of these well-known sequences had the groups previously listed on their mini-whiteboards?

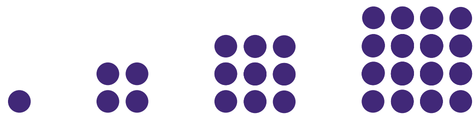
As students suggest a correct pairing, ask questions:

- ### Odd/Even numbers

  How is the sequence generated? Is this a term-to-term rule? (yes) How could I work out the 100th term without writing out all 100 terms? Is there a rule relating to the position of the term I am looking for? (The rules re $2n + 1$ and $2n$ respectively, but it may be too early to introduce the algebra. Instead students might say 'you double the term number and add 1')

- ### Square numbers

  How is the sequence generated? Why are the numbers called 'square'? Could we represent the sequence in diagram form? (A sequence of dots as shown would be one suitable way):



- ### Triangular numbers

  How is the sequence generated? Why are the numbers called 'triangular'? Could we represent the sequence in diagram form? (A sequence of dots as shown would be one suitable way):

- ### Cube numbers

  How is the sequence generated? Why are the numbers called 'cube'? How could they be represented? (The image on http://www.bbc.co.uk/bitesize/ks3/maths/algebra/number_patterns/revision/4/ illustrates the reason)

- ### Fibonacci numbers

  How is the sequence generated? Is there a link between the term number and the terms of the sequence? (not an easy one! See Skills Builder 3 later).

Now display the definitions below on the board and discuss the words ‘progression’, ‘arithmetic’ and ‘geometric’. Ask students to sort the six sequences on the cards into categories:

- Arithmetic sequences
- Geometric sequences
- Neither

An arithmetic progression or arithmetic sequence changes by **adding** or **subtracting** the same amount each time,

eg 3, 6, 9, 12, …. or 12, 10, 8, 6, ….

A geometric progression or geometric sequence changes by **multiplying** or **dividing** by the same amount each time,

eg 3, 6, 12, 24, …. or 100, 50, 25, 12.5, …

Sequences can be either arithmetic or geometric or neither.

Review the answers:

- Arithmetic – odd numbers, even numbers
- Geometric – none of these
- Neither – Fibonacci, triangular, square, cube

Ensure that students are clear that for a sequence to be geometric there must be a common multiplier between each pair of consecutive terms, which is not true for any of these sequences.

Developing Understanding 2

This section introduces the idea of term-to-term and position-to-term rules.

Display the following box on the board:

Sequences can be defined using a **term-to-term** rule.

Each term is generated from the previous term using the rule.

What term-to-term rules might we use?

Give students a few minutes to generate some sequences on their mini-whiteboards using as many different types of rules as possible. Ask students to share their sequences with the class and explain their rules. Ask students to consider if their sequences are:

- Arithmetic
- Geometric
- Fibonacci type (a sequence starting with any two numbers but formed in the same way as the Fibonacci numbers)

Explain that arithmetic sequences are often called linear sequences as they make a straight line on a graph when the term is plotted on the y-axis and the term number is plotted on the x-axis. It would be beneficial to demonstrate this for one of the arithmetic sequences that a student has generated.

Next, display the following box on the board:

A **position-to-term** rule describes each term of the sequence in relation to its position in the sequence.

This rule can be given in words or in algebra.

The position is usually denoted by the letter n so if we want the 10th term, we would put $n = 10$ into a formula.

For example 2, 4, 6, 8, 10, …. is obtained by doubling the term number

Term 1 = 2×1

Term 2 = 2×2

Term 3 = 2×3

Term 4 = 2×4 etc.

So a formula for the nth term would be $2 \times n$ or $2n$

This can be seen more clearly by looking at a table:

Position number	1	2	3	4	5
Term	2	4	6	8	10

Depending on the group, you might want to reveal the information a section at a time, so that students can think about the rule and deduce the rule in words and/or algebra.

Ask students:

- What is the 17th term of this sequence?
- Is 103 in this sequence? Why / why not?
- What is the position of the term 84?

The answers are:

- 17th term is $2 \times 17 = 34$
- 103 is not in the sequence as all the numbers are even.
  The term number for 103 would have to be $103 \div 2 = 51.5$ and this is not possible as position numbers have to be whole numbers
- $84 \div 2 = 42$ so it is the 42nd term.

Now display the following box and ask students to determine the position-to-term rule for each of the sequences. Discourage students from doing the (easier) term-to-term rule, perhaps by asking what the 500th term would be. Students should be encouraged to recognise the fact that having a position-to-term rule means any term or position number can be easily calculated, without the need to work out lots and lots of terms in the sequence.

Work out the position-to-term rule in words for each of these sequences:

- 5, 10, 15, 20, ….
- 13, 26, 39, 52, …
- 2, 5, 8, 11, …..
- 3, 9, 27, 81, ….
- 1, 2, 4, 8, ….
- 2, 3, 5, 9,…
- 1, 4, 5, 9, 14, …

Can you determine the algebra for any of these position-to-term rules?

The first three sequences are quite accessible and all students will hopefully be able to explain the position-to-term rule for these. Encourage students to write out what is happening term by term or in a table (as shown in the example on the previous page), if necessary.

The other sequences are more difficult and are likely to be appropriate for more able students. They could be replaced by further linear sequences if preferred.

It is unlikely that students will be able to generate the algebra for many of the sequences, but for very able students it would be useful extension task. Generating the algebra is not a key focus at this stage.

When reviewing the answers, ask students additional questions to check understanding of the rules, For example

- What is the 50th term in the sequence?
- What is the 37th term in the sequence?
- Is the number 60 (for example) in the sequence? How do you know?
- Can the rule be written easily in algebra?
- Which sequences are arithmetic? (the first three)
- Which sequences are geometric? (fourth and fifth)
- Which sequences are Fibonacci type? (the last sequence)

The answers are:

5, 10, 15, 20, ….	'multiply position number by 5'	$5n$
13, 26, 39, 52, ….	'multiply position number by 13'	$13n$
2, 5, 8, 11, ….	'multiply position number by 3 then subtract 1'	$3n - 1$
3, 9, 27, 81, ….	'3 to the power of the position number'	3^n
1, 2, 4, 8, ….	'2 to the power of the position number minus 1'	2^{n-1}
2, 3, 5, 9,….	'same as the previous sequence then add one'	$2^{n-1} + 1$
1, 4, 5, 9, 14, ….	'add together the previous two terms' (starting with 1 and 4 as the first two terms)	

To finish this section, display the following statements on the board, one at a time. Ask students to write down on their mini-whiteboards a sequence that meets the given rule – they might include four or five terms to show clearly the pattern that they intend to generate. When students hold up their whiteboards, identify any themes or common misconceptions – see some prompt questions below.

1. An arithmetic sequence with a difference of two between the terms
2. A Fibonacci type sequence starting with the number 6
3. A geometric sequence in which the next term is double the previous term
4. A sequence based on the square numbers – use one operation from +, –, × or ÷
5. A sequence based on the triangular numbers - use one operation from +, –, × or ÷
6. An arithmetic sequence involving fractions
7. A geometric sequence involving division
8. A sequence based on cube numbers – use one operation from +, –, × or ÷

Possible themes to draw out through questioning relating to the each of the eight given sequences:

1 Must this be an increasing sequence?
Does it matter which number we start with?
What do the formulae for these sequences have in common? (all contain 2*n*)

2 How many sequences are possible?
(an infinite number, it depends on the second number chosen)

3 Must this be an increasing sequence?
(No – starting with a negative number would make it decreasing)

4 (For example, add 1 to each term or multiply each term by 2, etc).
Ask individual students if they can identify what another student has done to the square numbers to obtain their sequence

5 Ask individual students if they can identify what another student has done to the triangular numbers to obtain their sequence

6 See if students can identify what rules other students have used to generate their sequences.
Is the difference a fraction?
Is the first term a fraction?
Must all terms be fractions?

7 Ask individual students if they can identify what number other students have divided by

8 Ask individual students if they can identify what another student has done to the cube numbers to obtain their sequence

Developing Understanding 3

In this section, the notation for recursive formulae is introduced. The formal use of notation might be too demanding for some students; however, sequences defined by recursive rules can also be described using words, and this would be suitable for all students. Therefore it might be necessary to adapt the materials below to suit the individual needs of your students.

Display the following information on the board. Students could work in pairs or small groups to determine the answers.

Lindsay is generating some sequences using a different term-to-term rule for each sequence.

1 1, 2, 4, 8, ….

2 5, 10, 20, 40, ….

3 2, 5, 11, 23, ….

4 1000, 500, 250, 125, ….

5 19, 17, 15, 13, ….

6 2, 4, 6, 10, 16, ….

Work out what the term-to-term rule is for each sequence.

The answers are:

1 Double the previous term starting with 1

2 Double the previous term, starting with 5

3 Double the previous term and add 1, starting with 2

4 Half the previous term, starting with 1000

5 Subtract 2 from the previous term starting with 19

6 Add together the two previous terms, starting with 2 and 4 as the first two terms

Discuss with students how the wording here can be quite long-winded and so some mathematical terminology has been introduced to summarise the information.

Display this box:

u_1	represents the first term of the sequence
u_2	represents the second term of the sequence
u_3	represents the third term of the sequence
u_n	represents the *n*th term of the sequence

Explain that we can write statements using these symbols for recursive sequences. They are generally of the form $u_{n+1} = f(u_n)$ ie the $(n + 1)$th term is written as a formula in terms of the previous term. (Do not express the recursive formulae on the board using function notation if students are not familiar with it or you think it would be off-putting).

Return to the first box showing Lindsay's sequences and display the worded answers alongside the sequences.

1	1, 2, 4, 8, ….	Double the previous term starting with 1
2	5, 10, 20, 40, ….	Double the previous term starting with 5
3	2, 5, 11, 23, ….	Double the previous term and add 1, starting with 2
4	1000, 500, 250, 125, ….	Half the previous term starting with 1000
5	19, 17, 15, 13, ….	Subtract 2 from the previous term starting with 19
6	2, 4, 6, 10, 16, ….	Add together the two previous terms, starting with 2 and 4 as the first two terms

In relation to question 1, ask students:

- How could we write the sentence 'To get the next term, double the previous term' using mathematical notation.
- How could we indicate that the sequence should start at 1?

Lead students through a discussion of possible ways of writing the information, correcting any misconceptions. Deduce that we can express the sequence as $u_{n+1} = 2u_n$ with $u_1 = 1$

Next, either support students in working through the notation for the remaining five questions, or allow them time to work through these themselves. Remind students that there will be two parts to the definition of the sequence – the recursive formula and the definition of the first term.

The answers are:

1 $u_{n+1} = 2u_n$ with $u_1 = 1$

2 $u_{n+1} = 2u_n$ with $u_1 = 5$

3 $u_{n+1} = 2u_n + 1$ with $u_1 = 2$

4 $u_{n+1} = \frac{u_n}{2}$ with $u_1 = 1000$

5 $u_{n+1} = u_n - 2$ with $u_1 = 19$

6 $u_{n+1} = u_n + u_{n-1}$ with $u_1 = 2, u_2 = 4$

The last one of these is quite demanding – students might need support in working out how to write 'the term before the last term' and in noting that we need two terms in order to start the sequence.

To complete this section, display the following box on the board. Students should complete the gaps so that each sequence is fully defined. When listing the terms of a sequence they should complete at least the first four terms. Students could be shown one question at a time and display the missing answer on a mini whiteboard, or alternatively they could work in pairs on the whole set of questions.

For students who find this difficult, instead of using the u_n notation, the sequences could be given to the students and they could be asked to describe the sequences in words, as above.

The last sequence is quite challenging, but will be suitable as an extension question for the more able students.

Display this box:

Complete the gaps in the following table:

Sequence	Worded definition	Recursive formula
1, 3, 9, 27, ….		
	Subtract 5 from the previous term, starting with 8	
		$u_{n+1} = \frac{u_n}{4}$ with $u_1 = 32$
	Double the previous term and then subtract 3, starting with 11	
		$u_{n+1} = (u_n)^2$ with $u_1 = 2$
	Add together the previous two terms, starting with 2 and 3 as the first two terms	
5, 11, 6, -5, -11, ….		

The answers are:

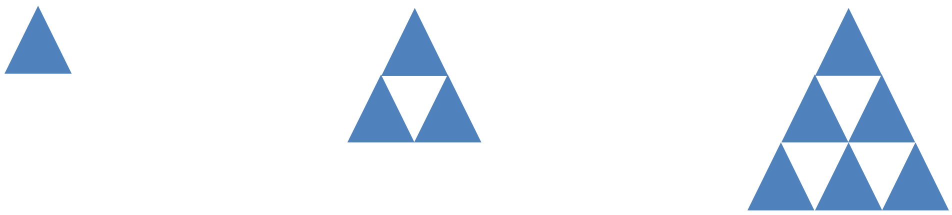
Sequence	Worded definition	Recursive formula
1, 3, 9, 27, ….	**Multiply the previous term by 3, starting with 1**	$u_{n+1} = 3u_n$ with $u_1 = 1$
8, 3, –2, –7, ….	Subtract 5 from the previous term, starting with 8	$u_{n+1} = u_n - 5$ with $u_1 = 8$
32, 8, 2, 0.5, ….	**Divide the previous term by 4, starting with 32**	$u_{n+1} = \frac{u_n}{4}$ with $u_1 = 32$
11, 19, 35, 67, ….	Double the previous term and then subtract 3, starting with 11	$u_{n+1} = 2u_n - 3$ with $u_1 = 11$
2, 4, 16, 256, ….	**Square the previous term, starting with 2**	$u_{n+1} = (u_n)^2$ with $u_1 = 2$
2, 3, 5, 8, ….	Add together the previous two terms, starting with 2 and 3 as the first two terms	$u_{n+2} = u_{n+1} + u_n$ with $u_1 = 2, u_2 = 3$
5, 11, 6, -5, -11, ….	**From the last term, subtract the term before that, starting with 5 and 11 as the first two terms**	$u_{n+2} = u_{n+1} - u_n$ with $u_1 = 5, u_2 = 11$

Skills Builder 1: Number sequences from diagrams

For each of the number patterns in the diagrams below:

- List the terms of the sequence
- Write down the next two terms (draw the patterns if you need to)
- Identify any other information you can about the sequence, eg explain the sequence in words, write down a formula, name the sequence, if it is arithmetic / geometric etc…

1



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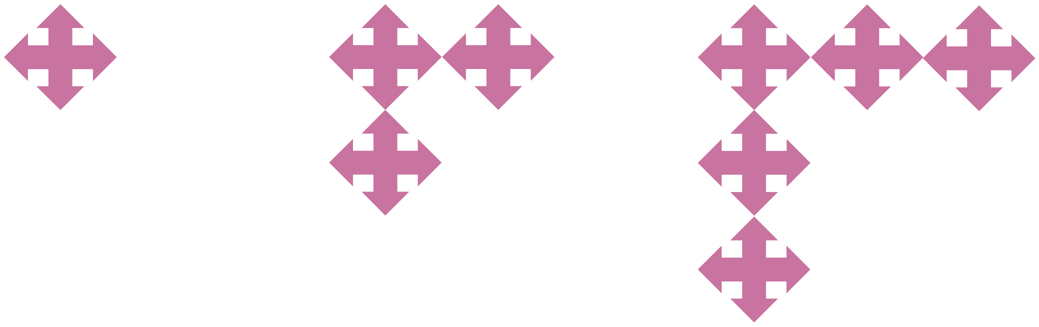

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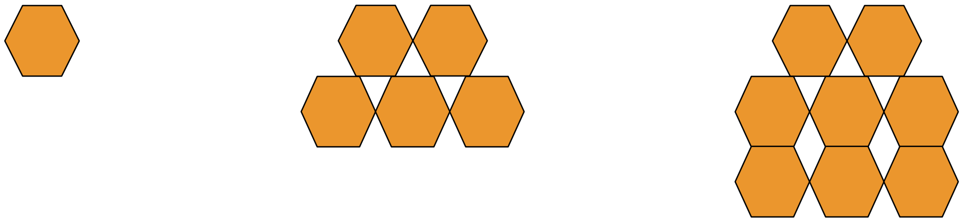

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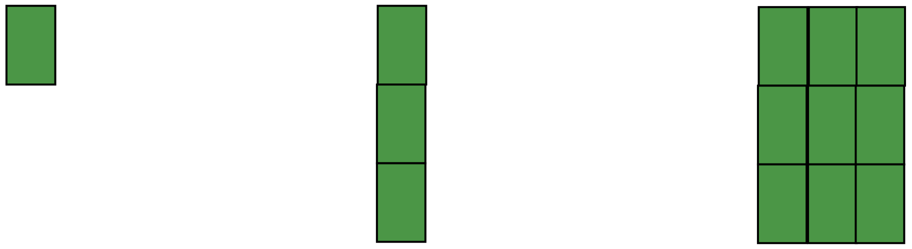

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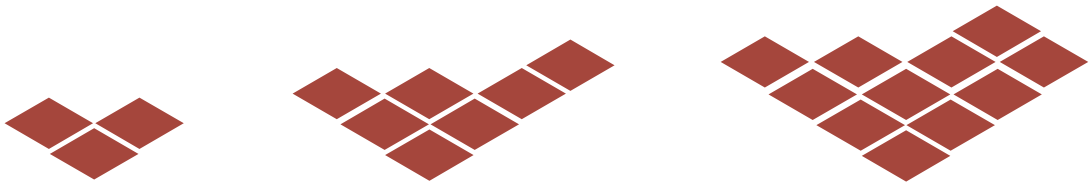

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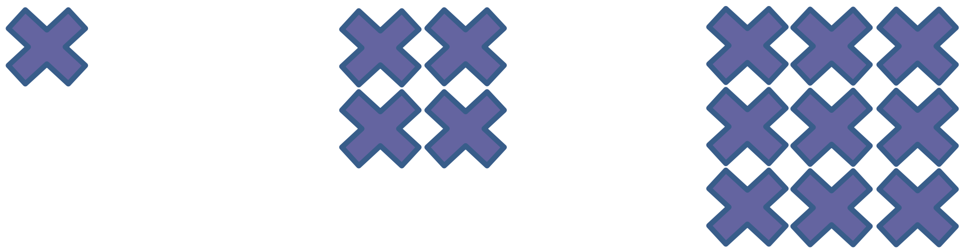

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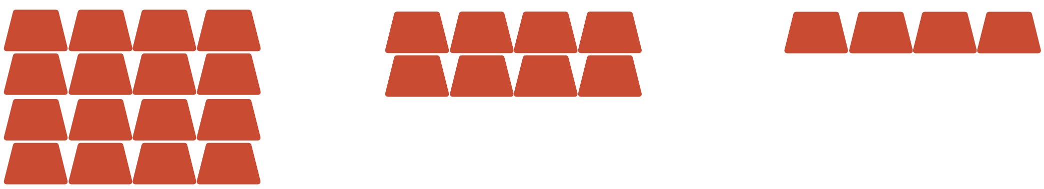

Skills Builder 2: Sequence rules

Term-to-term rules

Complete the table so that each sequence is defined as

- a list of numbers (the first four terms);
- using a term-to-term rule in words;
- using a recursive formula.

State also is the sequence is arithmetic, geometric or neither.

Sequence	Term-to-term rule (words)	Recursive formula	Arithmetic, geometric or neither
2, 9, 16, 23, ….			
	Add 5 each time starting with 3		
		$u_{n+1} = 3u_n$ with $u_1 = 6$	
	Divide by 5 each time starting with 1 000 000		
		$u_{n+2} = u_{n+1} + u_n$ with $u_1 = 4$, $u_2 = 8$	
1, 4, 16, 64, ….			
3, 7, 15, 31, ….			
–3, 6, –12, 24, ….			
		$u_{n+1} = \frac{1}{5}u_n$ with $u_1 = 125$	
	Multiply by 2 and then add 5 starting with 0.4		

Position-to-term rules

Find a position-to-term rule for each of the following sequences. Parts (a) to (g) are linear; parts (h) to (l) are quadratic (ie they are based on the square numbers).

(a) 3, 10, 17, 24, ….	(b) 2, 5, 8, 11, ….	(c) 9, 14, 19, 24, ….
(d) 5, 7, 9, 11, ….	(e) 4, 5, 6, 7, ….	(f) 4, 14, 24, 34, ….
(g) 13, 26, 39, 52, ….	(h) 1, 4, 9, 16, ….	(i) 2, 8, 18, 32, ….
(j) 0, 3, 8, 15, ….	(k) 3, 12, 27, 48, ….	(l) 6, 9, 14, 21, ….

Skills Builder 3: Harder sequences

1 For each of the sequences in the table, write down the next two terms and a suitable formula to represent the sequence – this could be either a recursive formula or algebra which represents a position-to-term rule.

Sequence	Next two terms	Formula
2, 4, 8, 16, ….		
3, 10, 17, 24, ….		
–10, –5, 0, 5, ….		
4, 7, 11, 18, ….		
2, 11, 26, 47, ….		
19, 18, 17, 16, …		
2, 16, 54, 128, …		

2 Find values of a and b such that the following statement is true:

A sequence of the form $u_{n+1} = au_n + b$ generates the terms 2, 11, 38, 119, ….

3 Find values of a and b such that the following statement is true:

A sequence of the form $u_{n+1} = \frac{u_n}{a} - b$ generates the terms 1405, 280, 55, 10, ….

4 The French mathematician Jacques Binet devised a formula for the Fibonacci numbers.

By substituting values into the formulae below, can you work out which one of the formulae is correct?

$$\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n}$$

$$\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{\sqrt{5}}$$

$$\frac{(1+\sqrt{5})^n + (1-\sqrt{5})^n}{2^n \times \sqrt{5}}$$

$$\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n \times \sqrt{5}}$$

Problem solving: Sequence sorting

The following cards represent sequences in a variety of different ways:

- In words
- As a list of numbers
- Using a position-to-term rule (formula)
- Using a term to term rule (recursive formula)

In pairs or small groups, students should consider the sequence on each card and sort them into three piles:

- Arithmetic sequences
- Geometric sequences
- Neither arithmetic nor geometric

Students score one point for each card they get in the correct column. They also score one bonus point for any additional fact they can state about the individual sequences. This might include:

- The common difference between terms
- The common multiplier between terms
- The name of the sequence
- An alternative representation of the sequence (eg, representing a sequence given as a list of numbers using a formula)

Depending on the ability of the group, make a judgement about whether you feel the bonus point should be awarded in each case.

Cards could be stuck to a large piece of paper and annotated with student comments.

This would form a useful classroom display.

Card 1 4, 15, 26, 37, ….	Card 2 $u_{n+1} = 4u_n$ with $u_1 = 3$
Card 3 1, 1, 2, 3, 5, ….	Card 4 Half the previous term starting with 700
Card 5 $u_n = 5n - 2$	Card 6 17, 11, 5, –1, ….
Card 7 $u_{n+1} = 4u_n$ with $u_1 = 2$	Card 8 $u_n = n^2$
Card 9 $u_n = 2n^3$	Card 10 2700, 900, 300, 100, ….
Card 11 $u_n = 20 - n$	Card 12 $u_{n+1} = 0.1u_n$ with $u_1 = 1$
Card 13 Add together the previous two terms, starting with 3 and –2 as the first two terms	Card 14 $u_{n+1} = u_n + (n + 1)$ with $u_1 = 1$

Problem solving 2: Sequence investigations

Investigation 1 – Pascal's triangle

The pattern of numbers below is called Pascal's triangle.

Row 0					1				
Row 1				1		1			
Row 2			1		2		1		
Row 3		1		3		3		1	
Row 4	1		4		6		4		1

Copy the pattern and look for connections between the entries on each row. Complete the next four rows using the patterns you have identified.

Look along the diagonals of Pascal's triangle – can you spot any patterns?

Add up the totals of the entries on each row – can you spot a pattern in the totals?

Investigation 2 – Fibonacci patterns

The pattern of numbers below is called Pascal's triangle.

Abi and Bennie are investigating patterns involving Fibonacci numbers.

Abi starts at the beginning of the sequence and adds up the squares of each number. The first four lines of her pattern are:

$1^2 = 1$

$1^2 + 1^2 = 2$

$1^2 + 1^2 + 2^2 = 6$

$1^2 + 1^2 + 2^2 + 3^2 = 15$ etc

Abi notices that the answers are also formed from some Fibonacci numbers.

Can you spot the pattern in the answers? Use your pattern to quickly work out the sum of the squares of the first seven Fibonacci numbers.

Bennie adds together the squares of two consecutive Fibonacci numbers and doubles his answer. For example:

$(1^2 + 2^2) \times 2 = 10$

$(2^2 + 3^2) \times 2 = 26$

$(3^2 + 5^2) \times 2 = 68$ etc

Bennie notices that the answers are also formed from some Fibonacci numbers.

Can you spot the pattern in the answers? (Hint: look at the Fibonacci numbers before and after the two being used on that row of the pattern).

Answers

Skills builder 1: Number sequences from diagrams

Students are unlikely to get all of the information for any particular sequence. Questions 4, 5 and 8 have more difficult position-to-term formulae and can easily be omitted – they are included here only for completeness.

1 1, 3, 6, 10, 15,… the triangular numbers

2 1, 3, 5, 7, 9, …. the odd numbers which have position-to-term formula $u_n = 2n - 1$ or recursive formula $u_{n+1} = u_n + 2$ with $u_n = 1$. This is an arithmetic sequence with common difference 2.

3 2, 5, 8, 11, 14, … sequence increases by 3 each time starting with 2. This is an arithmetic sequence with common difference 3. It has position-to-term formula $u_n = 3n - 1$ or recursive formula $u_{n+1} = u_n + 3$ with $u_n = 2$.

4 1, 3, 9, 27, 81, … sequence multiplies by 3 each time starting with 1. This is a geometric sequence with position-to-term formula $u_n = 3^{n-1}$ (may be difficult for students to spot) or recursive formula $u_{n+1} = 3u_n$ with $u_1 = 1$

5 2, 4, 8, 16, 32, … sequence multiplies by 2 each time starting with 2. This is a geometric sequence with position-to-term formula $u_n = 2^n$ or recursive formula $u_{n+1} = 2u_n$ with $u_1 = 2$
This is the sequence of powers of 2.

6 3, 7, 11, 15, 19, … sequence increases by 4 each time starting with 3. This is an arithmetic sequence with position-to-term formula $u_n = 4n - 1$ or recursive formula $u_{n+1} = u_n + 4$ with $u_1 = 3$

7 1, 4, 9, 16, 25, … the square numbers with position-to-term formula $u_n = n^2$

8 16, 8, 4, ,2, 1, …. Sequence halves each time starting with 16. This is a geometric sequence with position-to-term formula $u_n = 32 \times \left(\frac{1}{2}\right)^n$ (very difficult) or recursive formula $u_{n+1} = \frac{1}{2}u_n$ with $u_1 = 16$

Skills builder 2: Sequence rules

Term-to-term rules

Sequence	Term-to-term rule (words)	Recursive formula	Arithmetic, geometric or neither
2, 9, 16, 23, ….	Add 7 each time starting with 2	$u_{n+1} = u_n + 7$ with $u_1 = 2$	Arithmetic
3, 8, 13, 18, ….	Add 5 each time starting with 3	$u_{n+1} = u_n + 5$ with $u_1 = 3$	Arithmetic
6, 18, 54, 162, ….	Multiply by 3 each time starting with 6	$u_{n+1} = 3u_n$ with $u_1 = 6$	Geometric
1 000 000, 200 000, 40 000, 8 000	Divide by 5 each time starting with 1 000 000	$u_{n+1} = \frac{u_n}{5}$ with $u_1 = 1\,000\,000$	Geometric
4, 8, 12, 20, ….	Add together the previous two terms each time, starting with 4 and 8	$u_{n+2} = u_{n+1} + u_n$ with $u_1 = 4$, $u_2 = 8$	Neither
1, 4, 16, 64, ….	Multiply by 4 each time starting with 1	$u_n = 4u_n$ with $u_1 = 1$	Geometric
3, 7, 15, 31, ….	Double the previous term and then add 1 starting with 3	$u_{n+1} = 2u_n + 1$ with $u_1 = 3$	Neither
–3, 6, –12, 24, ….	Multiply the previous term by –2 starting with –3	$u_{n+1} = -2u_n$ with $u_1 = -3$	Geometric
125, 25, 5, 1, ….	Multiply the previous term by $\frac{1}{5}$ starting with 125	$u_{n+1} = \frac{1}{5}u_n$ with $u_1 = 125$	Geometric
0.4, 5.8, 16.6, 38.2, ….	Multiply by 2 and then add 5 starting with 0.4	$u_{n+1} = 2u_n + 5$ with $u_1 = 0.4$	Neither

Position-to-term rules

(a) 3, 10, 17, 24, …. $7n - 4$	(b) 2, 5, 8, 11, …. $3n - 1$	(c) 9, 14, 19, 24, …. $5n + 4$
(d) 5, 7, 9, 11, …. $2n + 3$	(e) 4, 5, 6, 7, …. $n + 3$	(f) 4, 14, 24, 34, …. $10n - 6$
(g) 13, 26, 39, 52, …. $13n$	(h) 1, 4, 9, 16, …. n^2	(i) 2, 8, 18, 32, …. $2n^2$
(j) 0, 3, 8, 15, …. $n^2 - 1$	(k) 3, 12, 27, 48, …. $3n^2$	(l) 6, 9, 14, 21, …. $n^2 + 5$

Skills builder 2: Harder sequence

1

Sequence	Next two terms	Formula
2, 4, 8, 16, ….	32, 64	$u_{n+1} = 2u_n$ with $u_1 = 2$
3, 10, 17, 24, ….	31, 28	$7n - 4$
–10, –5, 0, 5, ….	10, 15	$5n - 15$
4, 7, 11, 18, ….	29, 47	$u_{n+2} = u_{n+1} + u_n$ with $u_1 = 4$, $u_2 = 7$
2, 11, 26, 47, ….	74, 107	$3n^2 - 1$
19, 18, 17, 16, …	15, 14	$20 - n$
2, 16, 54, 128, …	250, 432	$2n^3$

2 $a = 3$, $b = 5$

3 $a = 5$, $b = 1$

4 The correct formula is on the bottom right-hand card

$\dfrac{(1+\sqrt{5})^n \times (1-\sqrt{5})^n}{2^n}$ $\dfrac{(1+\sqrt{5})^n \times (1-\sqrt{5})^n}{\sqrt{5}}$

$\dfrac{(1+\sqrt{5})^n \times (1-\sqrt{5})^n}{2^n \times \sqrt{5}}$ $\dfrac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n \times \sqrt{5}}$

$$\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n \times \sqrt{5}}$$

Problem solving 1: Sorting sequences

The arithmetic sequences are:

- Card 1 adding 11 each time, gives the formula $u_n = 11n + 7$
- Card 5 gives the sequence 3, 8, 13, 18, …
- Card 6 subtracting 6 each time, gives the formula $u_n = 23 - 6n$

The geometric sequences are:

- Card 2 the multiplier is 4 giving the sequence 3, 12, 36, 108, …
- Card 4 multiplier is $\frac{1}{2}$ giving the sequence 700, 350, 175, 87.5, …
- Card 10 multiplier is $\frac{1}{3}$ which has the formula $u_{n+1} = \frac{u_n}{3}$ with $u_1 = 2700$
- Card 12 multiplier is 0.1 which gives the sequence 1, 0.1, 0.01, 0.001, …

The sequences that are neither arithmetic nor geometric are:

- Card 3 the Fibonacci numbers
- Card 7 multiply by 4 and subtract 1, starting with 2 giving the sequence 2, 7, 27, 107, ...
- Card 8 the square numbers 1, 4, 9, 16, …
- Card 9 the sequence 2, 16, 54, 128, … which is double the sequence of cube numbers
- Card 13 this would give the sequence 3, –2, 1, –1, … and the formula would be $u_{n+1} = u_n + u_{n+1}$ with $u_1 = 3$, $u_2 = -2$
- Card 14 the sequence is 1, 3, 6, 10, … which is the triangular numbers

Problem solving 2: Sequence investigations

Investigation 1: Pascal's triangle

Row 0	**1**								
Row 1	**1**	**1**							
Row 2	**1**	**2**	**1**						
Row 3	**1**	**3**	**3**	**1**					
Row 4	**1**	**4**	**6**	**4**	**1**				
Row 5	**1**	**5**	**10**	**10**	**5**	**1**			
Row 6	**1**	**6**	**15**	**20**	**15**	**6**	**1**		
Row 7	**1**	**7**	**21**	**35**	**35**	**21**	**7**	**1**	
Row 8	**1**	**8**	**28**	**46**	**70**	**46**	**28**	**8**	**1**

The diagonals identified, from 'top to bottom' are: the positive integers, the triangular numbers, the pyramidal or tetrahedral numbers (these can be identified visually on the website http://www.mathsisfun.com/tetrahedral-number.html and are formed by adding consecutive triangular numbers, eg 1 + **3** = 4, 4 + **6** = 10, 10 + **10** = 20, etc…..

Adding up the totals for each row gives the totals 1, 2, 4, 8, 16, …. which are the powers of 2.

Specifically, for row n the formula for the total is 2^n

Investigation 2: Fibonacci patterns

Abi's pattern is:

$$1^2 = 1 = 1 \times 1$$

$$1^2 + 1^2 = 2 = 1 \times 2$$

$$1^2 + 1^2 + 2^2 = 6 = 2 \times 3$$

$$1^2 + 1^2 + 2^2 + 3^2 = 15 = 3 \times 5$$

So the sum of the squares of the first n Fibonacci numbers is equal to the product of the nth and $(n + 1)$th Fibonacci numbers.

In algebra this can be written as:

$$F_1^{\,2} + F_2^{\,2} + F_3^{\,2} + \ldots .\ F_n^{\,2} = F_n \times F_{n+1}$$

although students are likely to find this difficult to express.

The sum of squares of the first seven Fibonacci numbers is therefore the product of the seventh and eighth Fibonacci numbers, ie 13 × 21 = 273.

Bennie's pattern is

$$(1^2 + 2^2) \times 2 = 10 = 1 + 9 \text{ or } 1 + 3^2$$

$$(2^2 + 3^2) \times 2 = 26 = 1 + 25 = 1 + 5^2$$

$$(3^2 + 5^2) \times 2 = 68 = 4 + 64 = 2^2 + 8^2$$

The answers are given by adding together the squares of the Fibonacci numbers before and after the two used on that row of the pattern.

In algebra this could be written as:

$$(F_n{}^2 + F_{n+1}{}^2) \times 2 = F_{n-1}{}^2 + F_{n+2}{}^2$$

although students are likely to find this difficult to express.